Multi-agent Inverse Transportation

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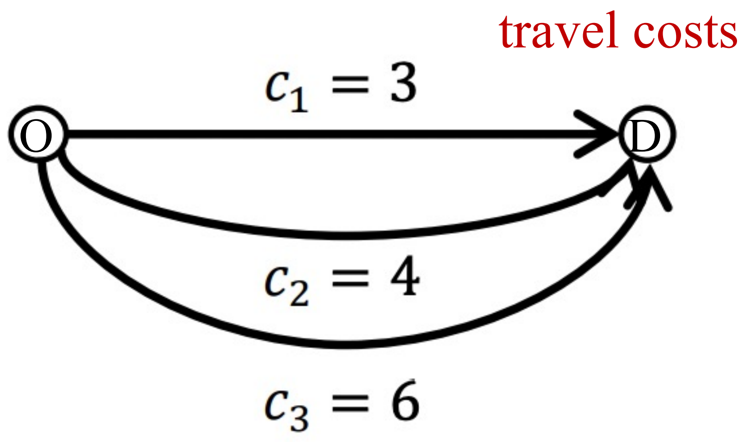
ORIE 7191 Project Presentation

Based on joint work with Susan Jia Xu, Joseph Chow, and Xintao Liu

Toy Network

- 400 homogeneous agents plan to travel from O to D.
- Which link will they choose?

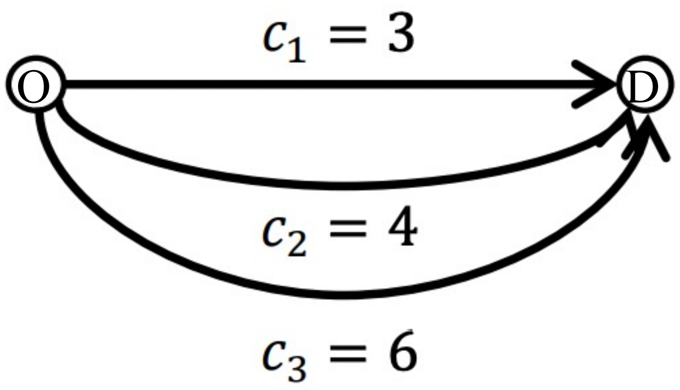


Capacitated Toy Network

- 400 homogeneous agents plan to travel from O to D.
- Which link will they choose?
- What if the links have capacity constraints?

travel costs

capacity constraints

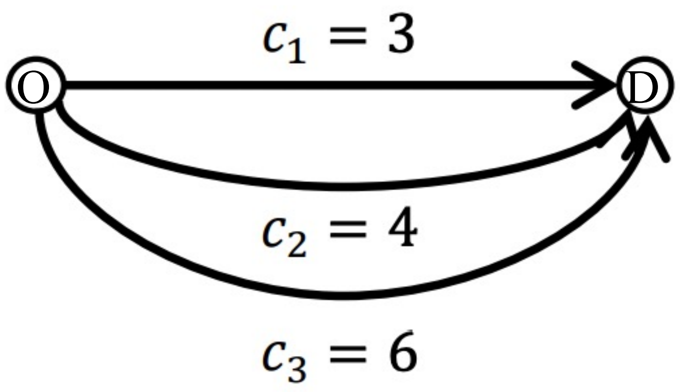


$u_1 = 100$

$u_2 = 200$

$u_3 = 300$

Minimum Cost Flow Problem



Capacity:

$u_1 = 100$
$u_2 = 200$
$u_3 = 300$

Demand:

$q_{OD} = 400$

$$\min z = c_1x_1 + c_2x_2 + c_3x_3$$

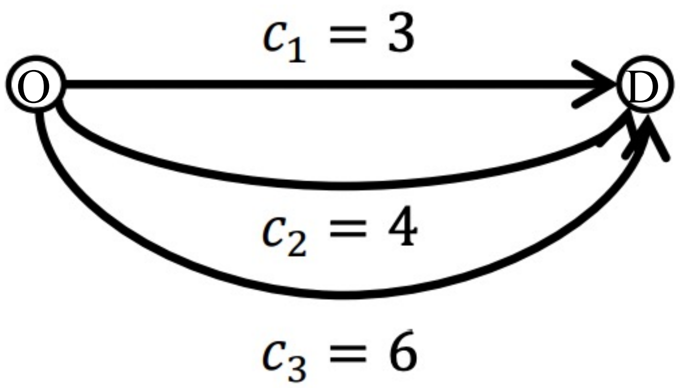
s.t.

$$x_1 + x_2 + x_3 = q$$
$$0 \le x_1 \le u_1$$
$$0 \le x_2 \le u_2$$
$$0 \le x_3 \le u_3$$

Solution: $x_1 = 100, \quad x_2 = 200, \quad x_3 = 100$

Inverse Problem

- What if we observe $x = (100, 200, 100)$
  - i.e., 100 agents choose link 1, 200 choose link 2, 100 choose link 3
  - Can we infer the values of u?



$u_1 = ?$

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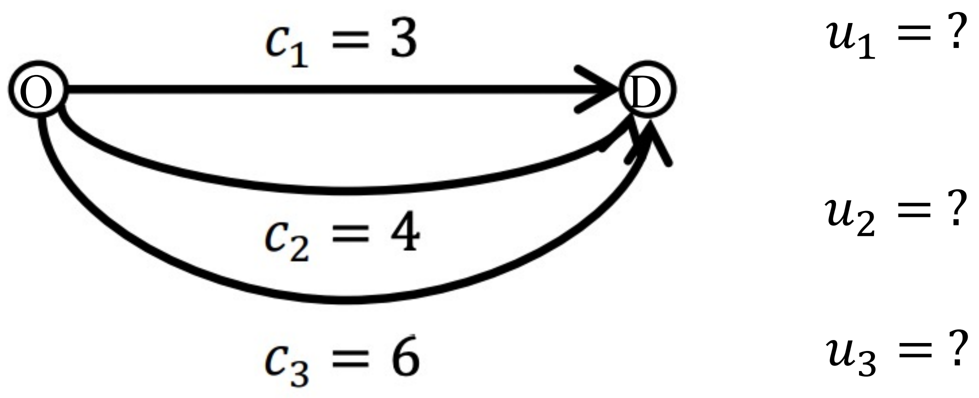
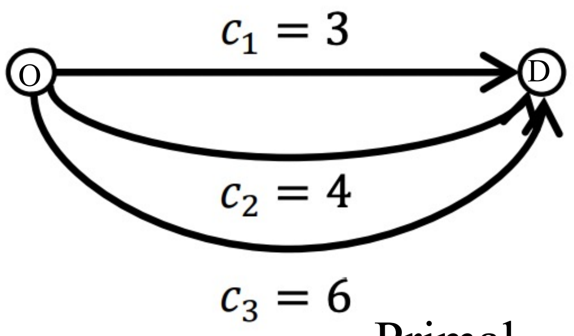
$u_3 = ?$

Inverse Problem

- What if we observe $x = (100, 200, 100)$
  - i.e., 100 agents choose link 1, 200 choose link 2, 100 choose link 3
  - Can we infer the values of u?
  - Hint: dual problem



Dual Problem



Dual variables:
π_i = node potential
w_j = link capacity dual (shadow) price

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Dual

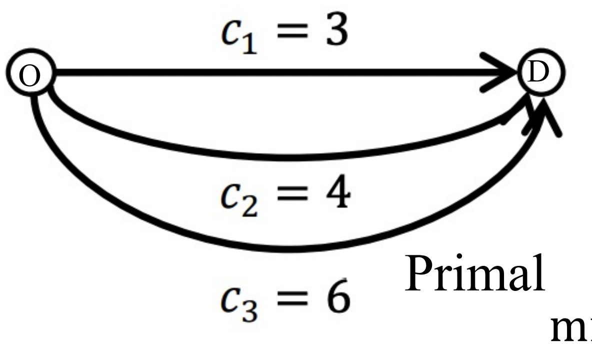
$$\max v = q\pi_O - q\pi_D - u_1 w_1 - u_2 w_2 - u_3 w_3$$

s.t.

$$\pi_O - \pi_D - w_1 \leq c_1$$
$$\pi_O - \pi_D - w_2 \leq c_2$$
$$\pi_O - \pi_D - w_3 \leq c_3$$
$$w_1, w_2, w_3 \geq 0$$

Solution: $w_1 = 3, \; w_2 = 2, \; w_3 = 0, \; \pi_O - \pi_D = 6$

Complementary Slackness



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Complementary slackness conditions: $w_i^*(x_i^* - u_i) = 0, \; (c_i + w_i^* - \pi_O^* + \pi_D^*)x_i^* = 0, \; \forall i$

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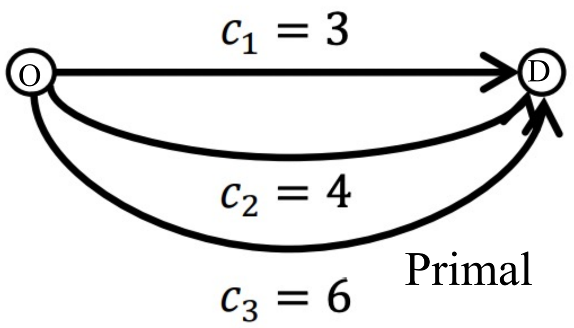

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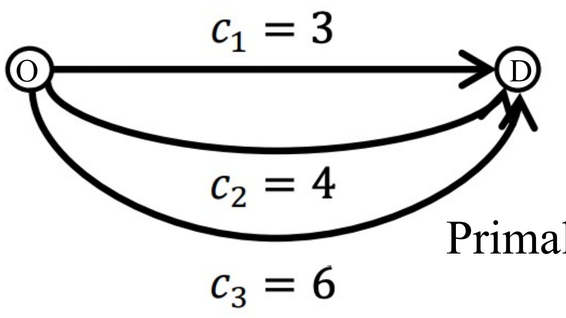
Complementary slackness conditions: $w_i^*(x_i^* - u_i) = 0, (c_i + w_i^* - \pi_O^* + \pi_D^*)x_i^* = 0, \forall i$
When there is no congestion on link i: $w_i^* = 0$
Impact of binding capacity on agents' route choices: $w_i^* > 0, x_i^* = u_i$

Indirect Approach



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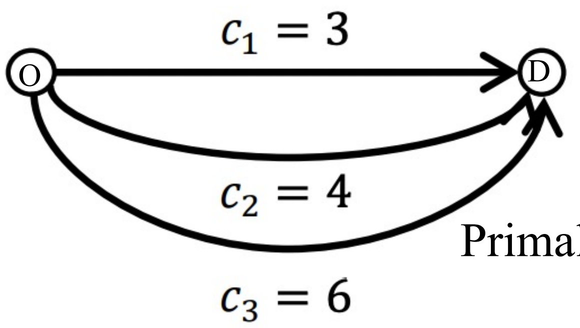
When there is no congestion on link i: $w_i^* = 0$
Impact of binding capacity on agents' route choices: $w_i^* > 0$, $x_i^* = u_i$
Indirect approach for inverse problem: instead of finding capacity, find the effects of the capacity and its interaction with agents – Find w!

Partial Dualization



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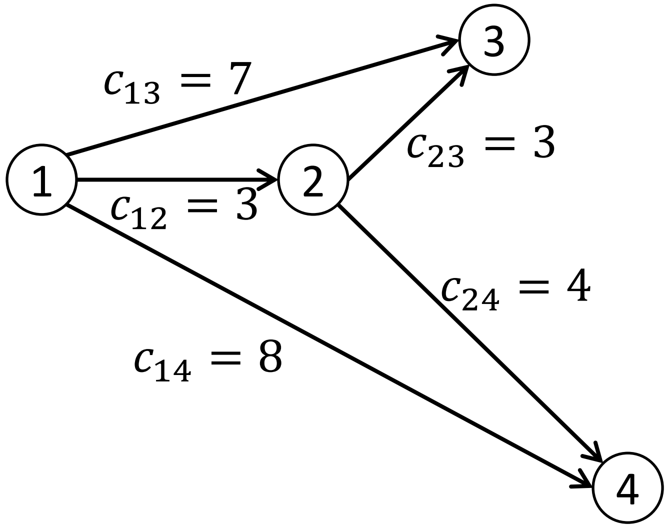
Solution: $w_1 = 3, \; w_2 = 2, \; w_3 = 0, \; \pi_O - \pi_D = 6$

Partial Dualization Theorem (Ahuja Ch17, p. 658). The flow variables $x^* = (100, 200, 100)$ solve the following equivalent uncapacitated shortest path problem:

$$\min \left\{ \sum_i (c_i + w_i)x_i : \; x_1 + x_2 + x_3 = q, x_i \ge 0 \right\}$$

Multicommodity Flow Problem



$q_{13} = 10$

$q_{14} = 20$

$K = \{(1,3), (1,4)\}$

$$\min z = \sum_{(i,j)} c_{ij} x_{ijk}$$

s.t.

$$\sum_{j \in N} x_{ijk} - \sum_{j \in N} x_{jik} = b_{ik}, \qquad \forall i \in N, k \in K$$

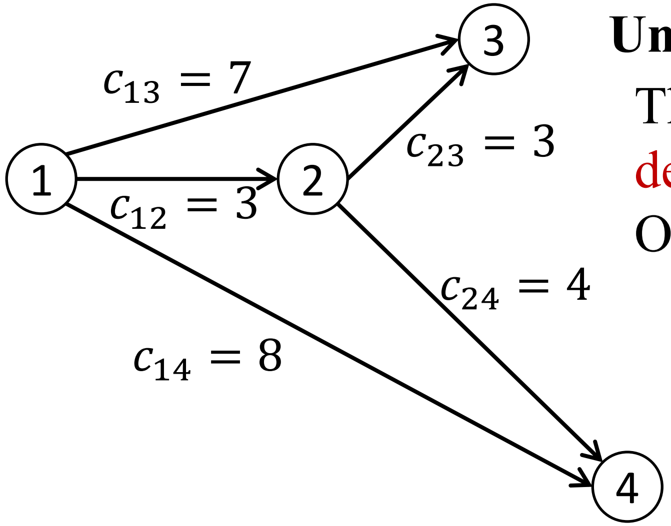
$$\sum_{k \in W} x_{ijk} \leq u_{ij}, \qquad \forall (i,j) \in A$$

$$x_{ijk} \geq 0$$

Theorem (Partial Dualization). Let x^*_{ijk} be optimal flows and let w^*_{ij} be optimal dual prices for the multicommodity flow problem. Then for each commodity k, the flow variables x^*_{ijk} solve the following uncapacitated minimum cost flow problem:

$$\min \left\{ \sum_{(i,j)} (c_{ijk} + w_{ij}) x_{ijk} : A x_k = b, x_{ijk} \geq 0 \right\}$$

Multi-agent Inverse Transportation



$q_{13} = 10$
$q_{14} = 20$
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Unconstrained shortest path problem

The restricted master problem can be decomposed into subproblems for OD pairs:

$$\min_{y} \phi = (c + w)^T y$$

Subject to

$$Ay = b$$ flow conservation

$$y_a \in \{0,1\}, \qquad a \in A$$

individual decision

Inverse shortest path problem

For each agent i, given observed route choice y_i^*, estimate the perception of the dual price $w_{a,i}$:

$$\min_{e_i, f_i, v_i} \phi_i^{-1} = e_i + f_i$$

Subject to

$$A^T \pi_i \leq c + \bar{w} - e_i + f_i$$ weak duality

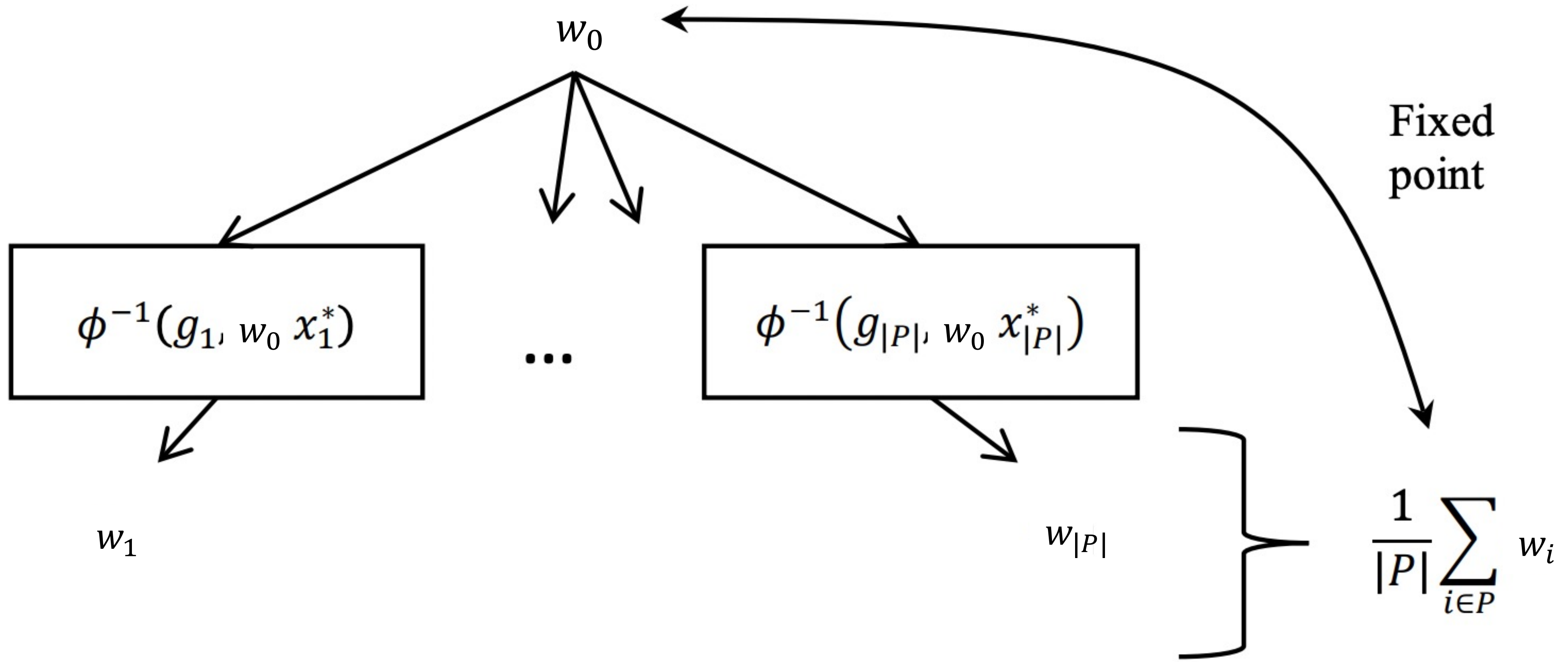
$$b^T \pi_i = (c + \bar{w} - e_i + f_i)^T y^*$$ strong duality

$$e_i - f_i \leq \bar{w}$$ capacity dual variable feasibility

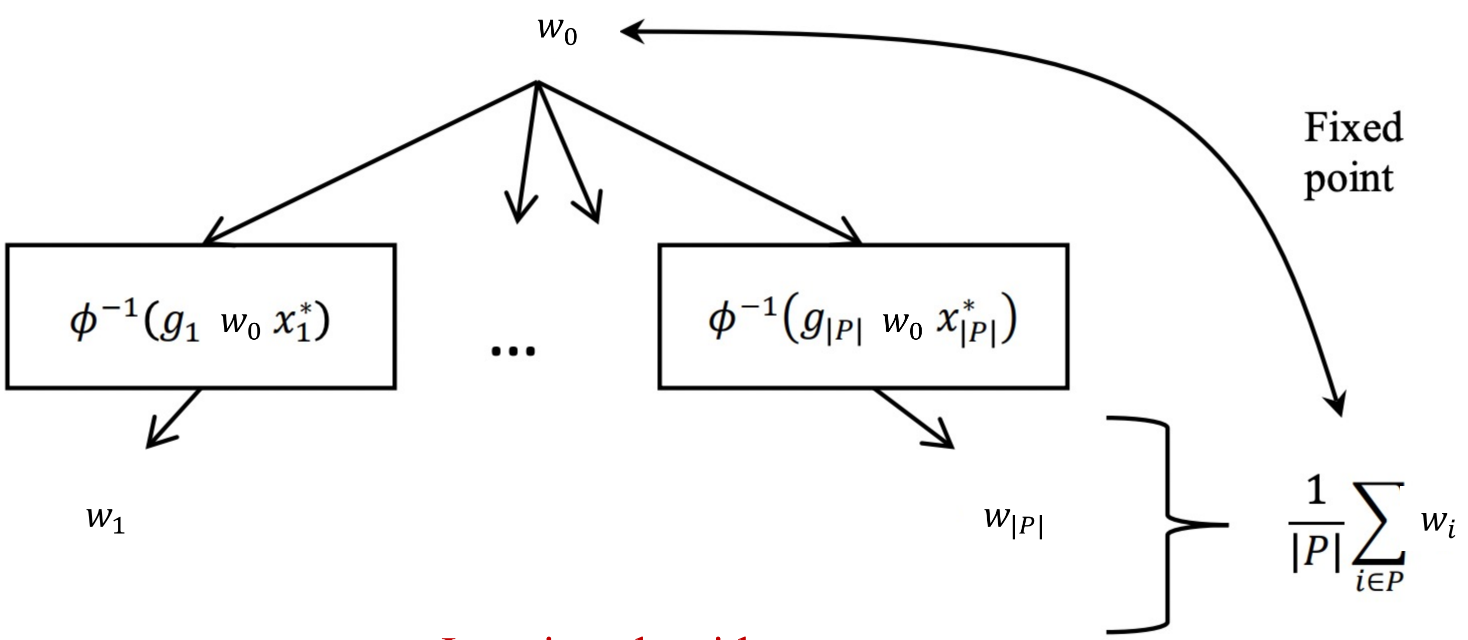
$$e_i, f_i \geq 0$$ non-negativity

$$w_i = \bar{w} - e_i + f_i$$ minimal perturbation from common prior

Multi-agent Inverse Transportation



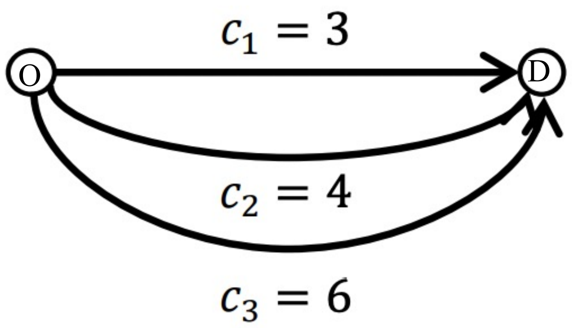
Iterative Algorithm



Iterative algorithm:

0. Given an initial common prior w_0^1 (e.g. previous update), and $n = 1$.
1. For each agent $i \in P$, solve an inverse shortest path problem with augmented link costs in Eq. (13), $w_i^n = \phi^{-1}(g_i, w_0^n, x_i^*)$.
2. Update common prior: $w_0^{n+1} = \frac{1}{|P|}\sum_{i \in P} w_i^n$. Set $n = n + 1$ and go to step 1 if $w_0^{n+1} \neq w_0^n$.

Iterative Algorithm



- Initiate: $w_0^1 = \{0,0,0\}$

Route choice observation:

$$x_1 = 100$$
$$x_2 = 200$$
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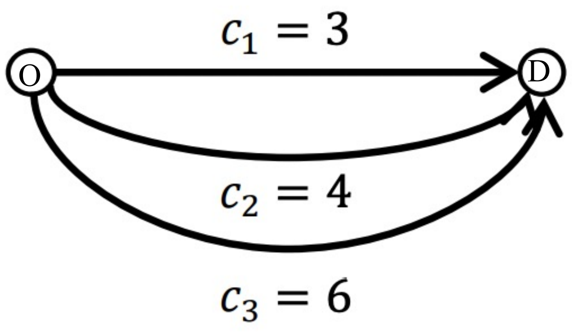

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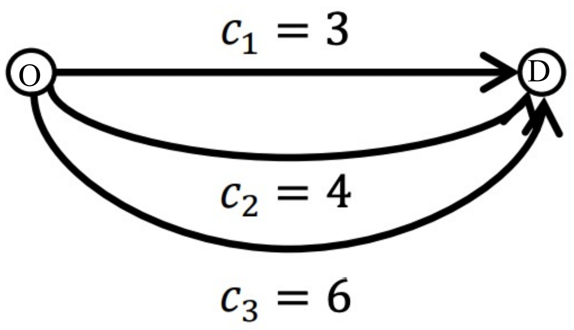

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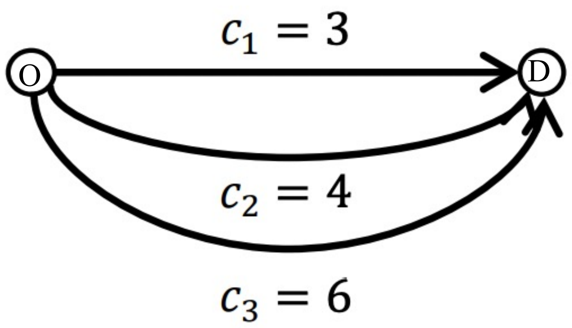

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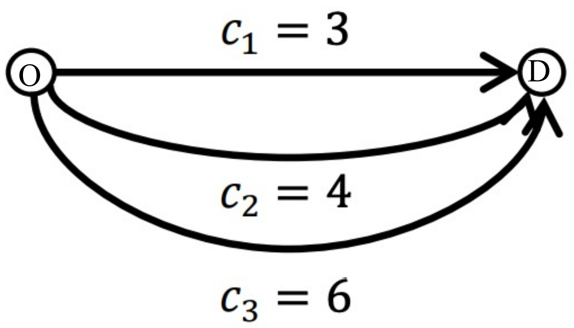

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- First iteration: $w_0^2 = \left\{\frac{5}{4}, \frac{1}{2}, 0\right\}$

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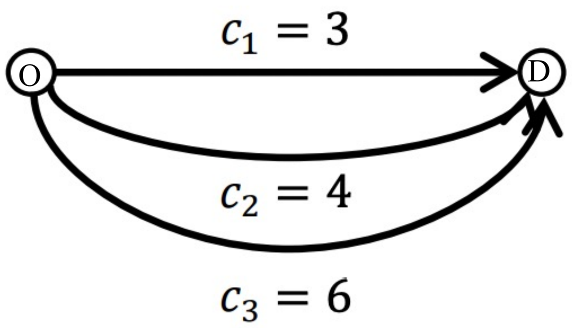

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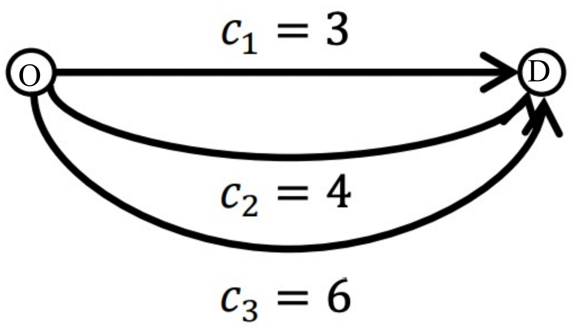

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- Converge to $w_0^* = w_1^* = w_2^* = w_3^* = \{3,2,0\}$

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Online Learning

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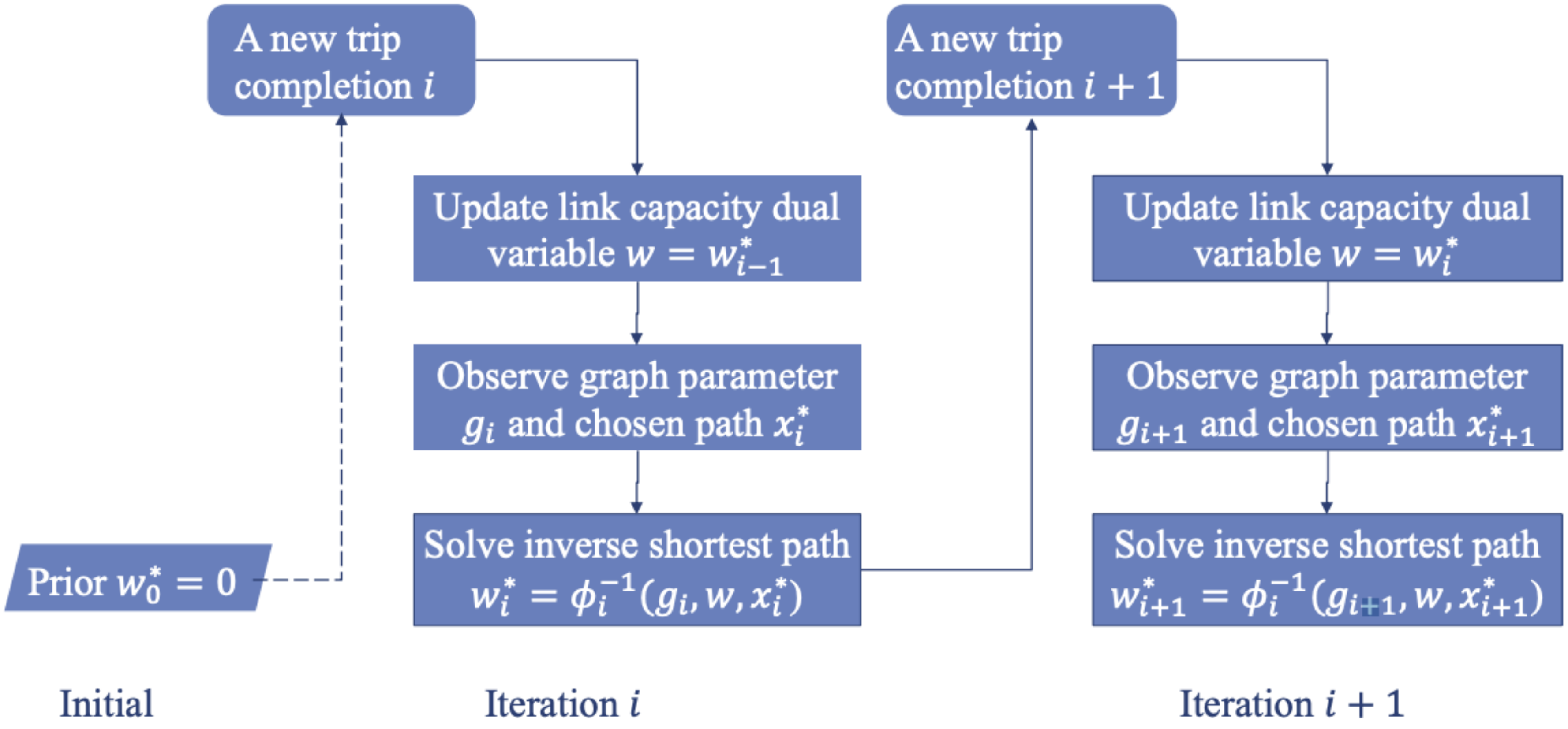
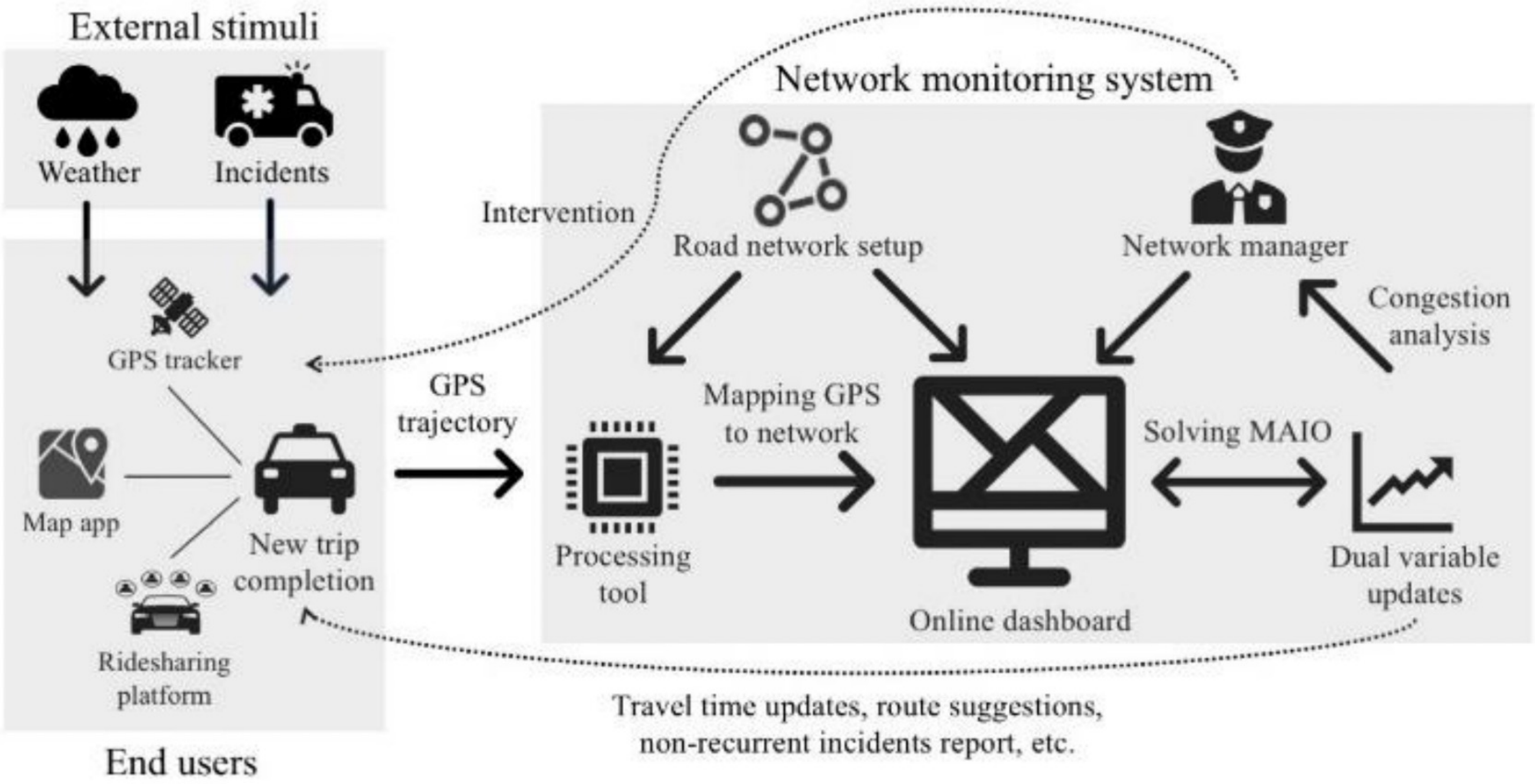


Network Monitoring Architecture



Validation Experiment Design

1. Initiate with values of link capacity dual variables equal to zero for all links in the study network.
2. Starting at 5:00AM, and every 5 minutes thereafter until 9:00AM,
   i. For all the trajectories that arrived in that period, identify OD pairs.
   ii. Run the path reconstruction algorithms to get real-time travelers' choices for each of the OD pairs (in this step, the traveler's choice is assumed as the shortest path).
   iii. Compare the predicted route and the actual route chosen.
   iv. Run MAIO to update the dual variables based on the reconstructed path.
   v. Compute the correlation between real travel times and estimated travel time.

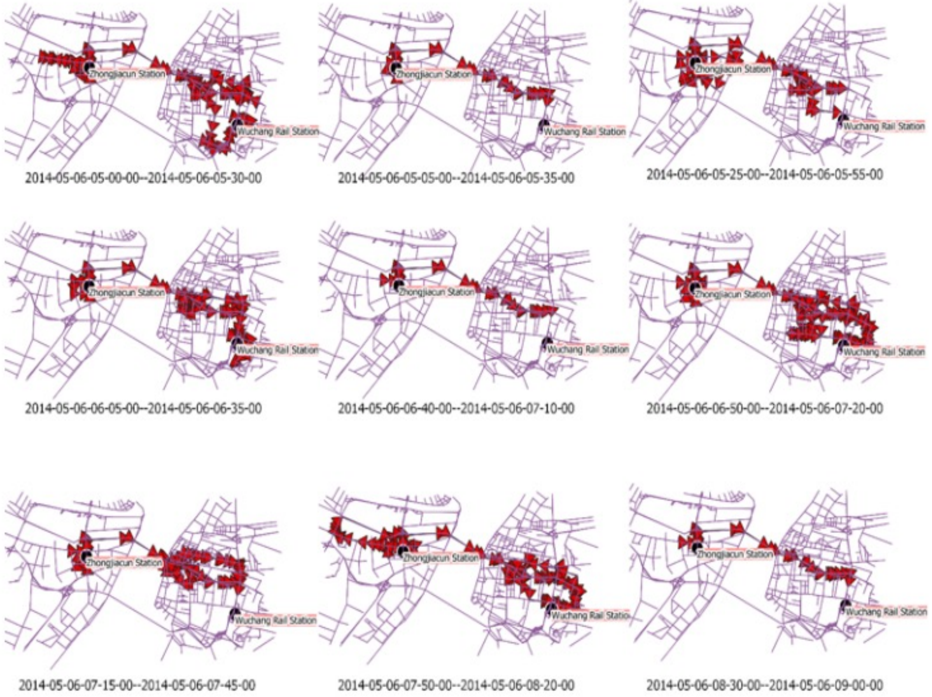
Case study: Wuhan Downtown

Wuhan, China

855 nodes
2833 links

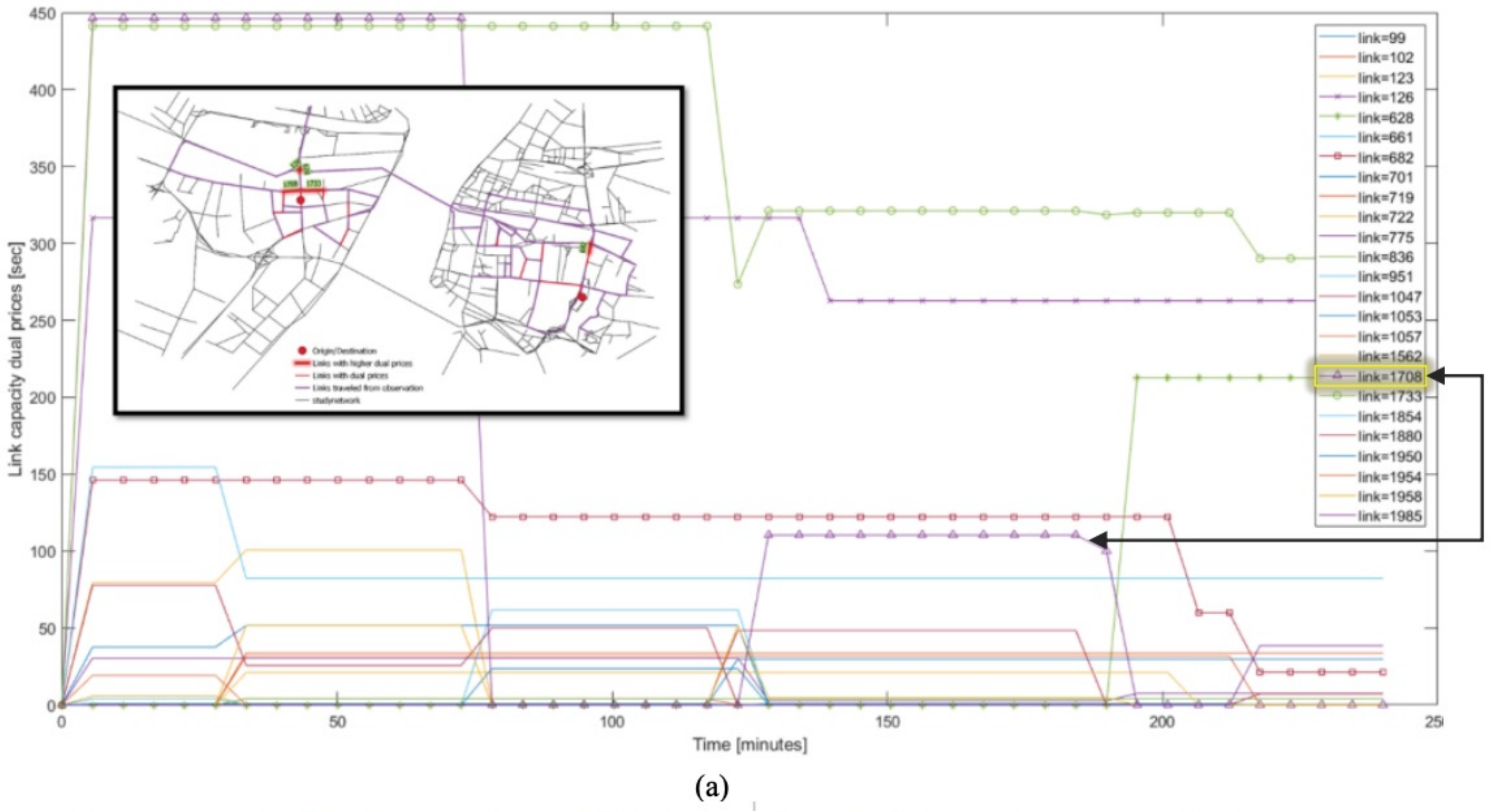
132 samples observed for OD 1
48 samples observed for OD 2

53 different path taken for OD 1
29 different path taken for OD 2



Samples of route diversions for one OD

Sensitivity to Network Changes



(a) Trajectories of link dual variables with one OD

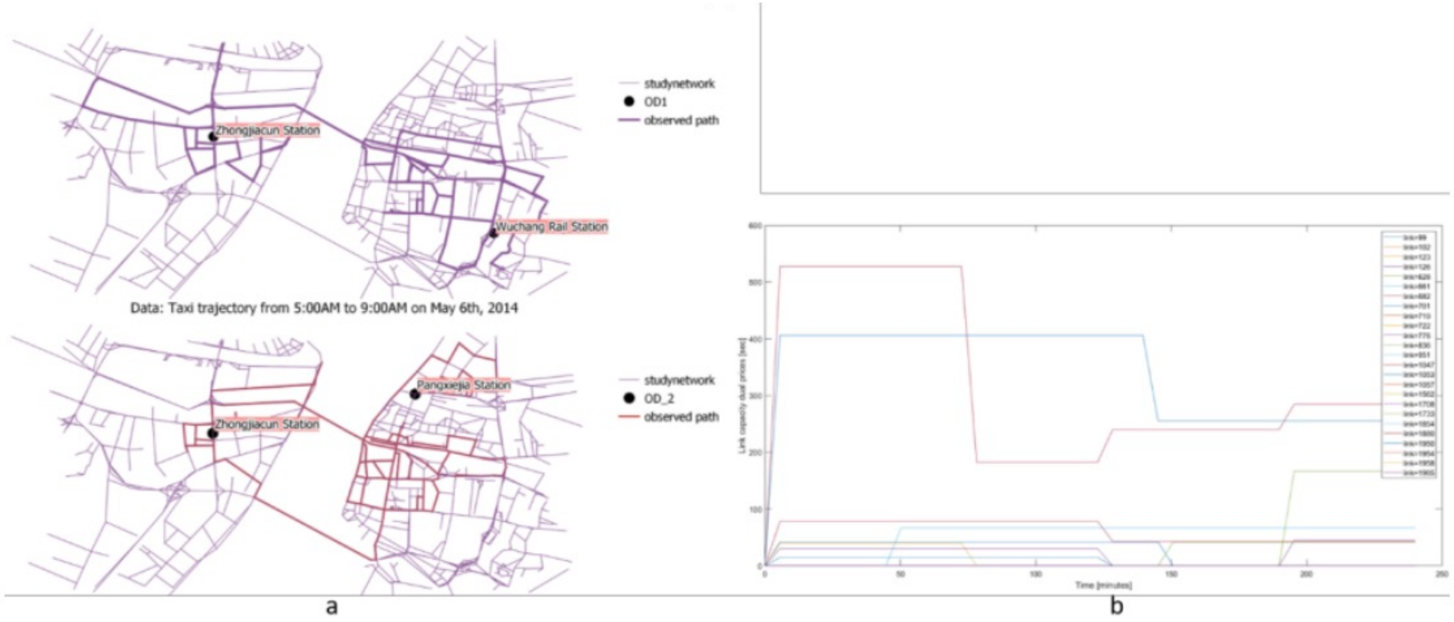
25 links with positive dual variables

Link 1708 highest, then drop to 0 ⇒ an incident in the earlier spike

Link 1733: sustained ⇒ heavy usage under recurrent congestion effect

MAIO can estimate dual variables (congestion effects) in real-word urban network

Sensitivity to Network Changes



(b)

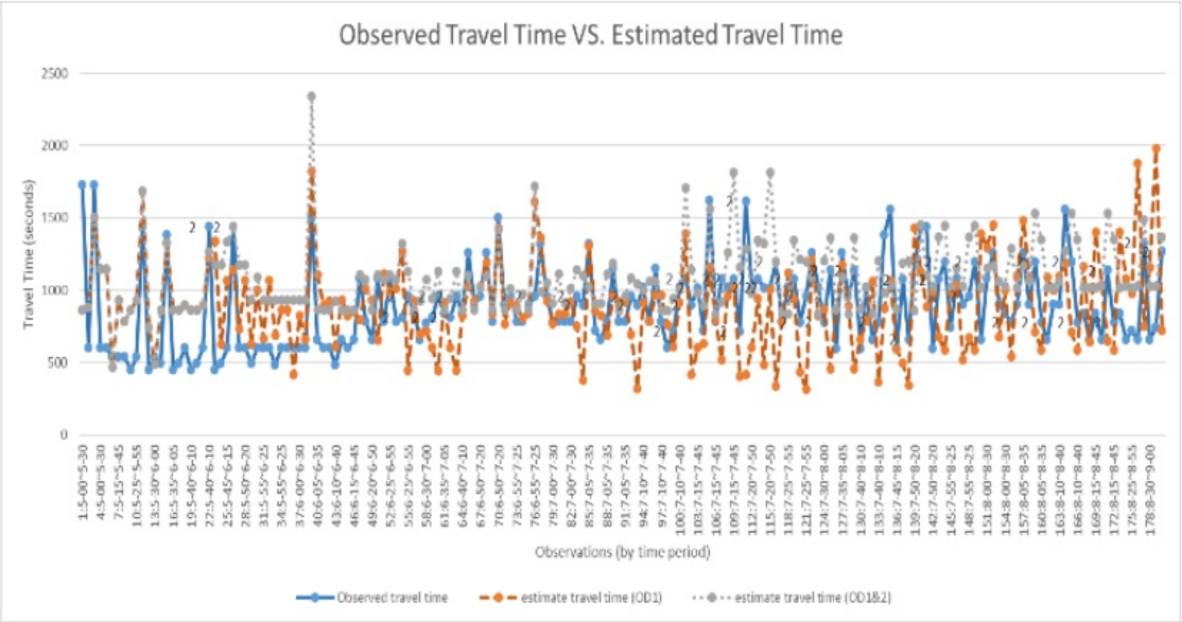
(b) Updated dual variables after adding new OD

Temporal profile of link dual variables look quite different

Link 1708 more accurate ⇒ effectiveness of MAIO depends on OD sampling

More route observations are considered, more information on dual variables is provided

Accuracy Improvement on Estimation



Estimated travel time and real travel time of 180 observed routes from single and two OD pairs

Estimated travel time is calculated as free flow travel time plus estimated dual variables on traveled links

Correlations between the observed and estimated travel time are 0.23 and 0.56

MAIO provides a good fit to real observations, even samples from only two ODs